THE COARSE BAUM-CONNES CONJECTURE FOR RELATIVELY HYPERBOLIC GROUPS

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Abstract. We study a group which is hyperbolic relative to a finite family of infinite subgroups. We show that the group satisfies the coarse Baum-Connes conjecture if each subgroup belonging to the family satisfies the coarse Baum-Connes conjecture and admits a finite universal space for proper actions. Especially, the group satisfies the analytic Novikov conjecture.

1. Introduction

Let X be a proper metric space. We say that X satisfies the coarse Baum-Connes conjecture if the following coarse assembly map μ_X of X is an isomorphism:

$$\mu_X \colon KX_*(X) \to K_*(C^*(X)).$$

If a countable group G equipped with a proper invariant metric satisfies the coarse Baum-Connes conjecture, and if G admits a finite G-simplicial complex which is a universal space for proper actions, then, by a descent principle, G satisfies the analytic Novikov conjecture. For details, see [17, Theorem 8.4] and also [7, Theorem 12.6.3].

There are several studies on the coarse Baum-Connes conjecture for relatively hyperbolic groups. Let G be a group which is hyperbolic relative to a finite family of infinite subgroups $\mathbb{P} = \{P_1, \dots, P_k\}$. Osin [14] showed that G has finite asymptotic dimension if each subgroup P_i has finite asymptotic dimension. Ozawa [15] showed that G is exact if each subgroup P_i is exact. Dadarlat and Guentner [2] showed that G is uniformly embeddable in a Hilbert space if each subgroup P_i is uniformly embeddable in a Hilbert space. Due to Yu's works [19][20], those results imply the coarse Baum-Connes conjecture for such groups.

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In the present paper, we show the following:

THEOREM 1.1. *Let G be a finitely generated group and $\mathbb{P} = \{P_1, \dots, P_k\}$ be a finite family of infinite subgroups. Suppose that $(G, \mathbb{P})$ is a relatively hyperbolic group. If each subgroup P_i satisfies the coarse Baum-Connes conjecture, and admits a finite P_i-simplicial complex which is a universal space for proper actions, then G satisfies the coarse Baum-Connes conjecture.*

We note that G admits a finite G-simplicial complex which is a universal space for proper actions (see Appendix B).

Here we summarize the proof of Theorem 1.1. Let $X(G, \mathbb{P}, \mathcal{S})$ be the augmented space obtained by attaching horoballs to the Cayley graph $\Gamma(G, \mathcal{S})$ along the left cosets of subgroups $P \in \mathbb{P}$ where $\mathcal{S}$ is a finite generating set (Definition 2.1 and Definition 2.2). Since $X(G, \mathbb{P}, \mathcal{S})$ is δ-hyperbolic, $X(G, \mathbb{P}, \mathcal{S})$ satisfies the coarse Baum-Connes conjecture. We fix an order on horoballs. Let X_n be a subspace obtained by removing the first $n-1$ horoballs from $X(G, \mathbb{P}, \mathcal{S})$ (Notation 5.1). By Mayer-Vietoris arguments, we show inductively that X_n satisfies the coarse Baum-Connes conjecture (Section 5.1). To study the coarse assembly map for $X_\infty = \bigcap X_n$, which is coarsely equivalent to G, we need to analyze the coarse K-homology of the projective limit. We might expect a so-called Milnor exact sequence

$$0 \to \varprojlim{}^1 KX_{p+1}(X_n) \to KX_p(X_\infty) \to \varprojlim KX_p(X_n) \to 0. \tag{1}$$

Unfortunately, (1) is not necessarily exact, in general. A simple counterexample is given by $Y_n = \mathbb{R} \setminus [-n, n]$. Thus we introduce a contractible space $EX(G, \mathbb{P})$. The following isomorphism (Proposition 3.1) is crucial to the proof of Theorem 1.1:

$$KX_*(X(G, \mathbb{P}, \mathcal{S})) \cong K_*(EX(G, \mathbb{P})).$$

Sections 2 and 3 are devoted to a proof of this isomorphism. For the projective limit of locally compact Hausdorff spaces, there is a Milnor exact sequence in K-homology (Section 5.2). Combining this with an exact sequence in K-theory of C^*-algebras (Proposition 5.3), we complete the proof.

2. COARSE K-HOMOLOGY OF THE AUGMENTED SPACE

Let G be a finitely generated group with a finite family of infinite subgroups $\mathbb{P} = \{P_1, \dots P_k\}$. Groves and Manning [4] introduced a space obtained by attaching "combinatorial horoballs" to G along the left cosets of subgroups $P \in \mathbb{P}$. Their construction is

suitable for Mayer-Vietoris arguments to compute the coarse K-homology of G in terms of that of $P \in \mathbb{P}$. We review the construction and study the coarse K-homology of the resulting space.

2.1. **The augmented space.**

DEFINITION 2.1. Let (P, d) be a proper metric space. *The combinatorial horoball* based on P, denoted by $\mathcal{H}(P)$, is the graph defined as follows:

(1) $\mathcal{H}(P)^{(0)} = P \times (\mathbb{N} \cup \{0\})$.
(2) $\mathcal{H}(P)^{(1)}$ contains the following two type of edges:
   (a) For each $l \in \mathbb{N} \cup \{0\}$ and $p, q \in P$, if $0 < d(p,q) \leq 2^l$ then there is a *horizontal edge* connecting (p, l) and (q, l).
   (b) For each $l \in \mathbb{N} \cup \{0\}$ and $p \in P$, there is a *vertical edge* connecting (p, l) and $(p, l+1)$.

Here $\mathbb{N}$ denotes the set of positive integers. We endow $\mathcal{H}(P)$ with the graph metric. For a closed subset $I \subset \mathbb{R}$, let $\mathcal{H}(P; I)$ denote the full subgraph of $\mathcal{H}(P)$ spanned by $P \times (I \cap (\mathbb{N} \cup \{0\}))$.

Let G be a finitely generated group with a finite family of infinite subgroups $\mathbb{P} = \{P_1, \dots P_k\}$. We take a finite generating set $\mathcal{S}$ for G. We assume that $\mathcal{S}$ is symmetrized, so that $\mathcal{S} = \mathcal{S}^{-1}$. We endow G with the left-invariant word metric $d_{\mathcal{S}}$ with respect to $\mathcal{S}$. We choose a sequence $g_1, g_2, \dots$ in G such that for each $r \in \{1, \dots, k\}$, the map $\mathbb{N} \to G/P_r : a \mapsto g_{ak+r} P_r$ is bijective. For $i = ak + r \in \mathbb{N}$, let $P_{(i)}$ denote a subgroup P_r. Thus the set of all cosets $\bigsqcup_{r=1}^{k} G/P_r$ is indexed by the map $\mathbb{N} \ni i \mapsto g_i P_{(i)}$. Each coset $g_i P_{(i)}$ has a proper metric d_i which is the restriction of $d_{\mathcal{S}}$. Let Γ be the Cayley graph of $(G, \mathcal{S})$. There exists a natural embedding $\psi_i \colon \mathcal{H}(g_i P_{(i)}; \{0\}) \hookrightarrow \Gamma$ such that $\psi_i(x, 0) = x$ for all $x \in g_i P_{(i)}$.

DEFINITION 2.2. *The augmented space* $X(G, \mathbb{P}, \mathcal{S})$ is obtained by pasting $\mathcal{H}(g_i P_{(i)})$ to Γ by ψ_i for all $i \in \mathbb{N}$. Thus we can write it as follows:

$$X(G, \mathbb{P}, \mathcal{S}) = \Gamma \cup \bigcup_{i \in \mathbb{N}} \mathcal{H}(g_i P_{(i)}).$$

We endow $X(G,\mathbb{P},\mathcal{S})$ with the graph metric. For positive integer N, set

$$\begin{aligned} X(N) &= \Gamma \cup \bigcup_{i\in\mathbb{N}} \mathcal{H}(g_i P_{(i)};[0,N]); \\ Y(N) &= \bigsqcup_{i\in\mathbb{N}} \mathcal{H}(g_i P_{(i)};[N,\infty)); \\ Z(N) &= \bigsqcup_{i\in\mathbb{N}} \mathcal{H}(g_i P_{(i)};\{N\}). \end{aligned}$$

Remark 2.3. The vertex set of $X(G,\mathbb{P},\mathcal{S})$, denoted by $X(G,\mathbb{P},\mathcal{S})^{(0)}$, can naturally be identified with the set of 2-tuple (x,t), where $x \in \bigsqcup_i g_i P_{(i)}$ and $t \in \mathbb{N}$, or $x \in G$ and $t = 0$. We endow $X(G,\mathbb{P},\mathcal{S})^{(0)}$ with the metric from the graph structure.

Definition 2.4. The pair $(G,\mathbb{P})$ is a relatively hyperbolic group if the augmented space $X(G,\mathbb{P},\mathcal{S})$ is δ-hyperbolic for some $\delta \geq 0$.

Remark 2.5. Groves and Manning [4, Theorem 3.25] show that the above definition is equivalent to other various definitions. See also [9].

2.2. **An anti-Čech system.** We form an anti-Čech system $\{\mathcal{U}(j)\}_j$ of $X(G,\mathbb{P},\mathcal{S})^{(0)}$ as follows: For $i \geq 1, (x,t) \in g_i P_{(i)} \times \mathbb{N}$ and $j \geq 1$, a column centered at (x,t) with the size j is

$$B((x,t),j) = \{(y,l) \in g_i P_{(i)} \times \mathbb{N} : d_{\mathcal{S}}(x,y) \leq 2^{t+j}, t \leq l \leq t+j\}.$$

For $x \in G$ and $j \geq 1$, a column centered at $(x,0)$ with the size j is

$$B((x,0),j) = \{(y,l) \in X(G,\mathbb{P},\mathcal{S})^{(0)} : d_{\mathcal{S}}(x,y) \leq 2^{j}, 0 \leq l \leq j\}.$$

The locally finite cover $\mathcal{U}(j)$ is made up of all those columns with size j, that is,

$$\mathcal{U}(j) = \{B((x,t),j) : (x,t) \in X(G,\mathbb{P},\mathcal{S})^{(0)}\}.$$

When $j \leq j'$, the map $\mathcal{U}(j) \to \mathcal{U}(j')$ is defined by sending $B((x,t),j)$ to $B((x,t),j')$.

2.3. **Mayer-Vietoris sequences.** Set $j_n = 3^n, N_n = 3^n + 1$ for $n \geq 0$. We introduce a decomposition of $\mathcal{U}(j_n)$ as follows:

$$\begin{aligned}
\mathcal{U}_n &= \mathcal{U}(j_n);\\
\mathcal{X}_n &= \{B \in \mathcal{U}(j_n) : B \cap X(N_n) \neq \emptyset\};\\
\mathcal{Y}_n &= \{B \in \mathcal{U}(j_n) : B \cap Y(N_n) \neq \emptyset\};\\
\mathcal{Z}_n &= \{B \in \mathcal{U}(j_n) : B \cap Z(N_n) \neq \emptyset\};\\
\mathcal{Z}_n^i &= \{B \in \mathcal{Z}_n : B \cap \mathcal{H}(g_i P_{(i)}) \neq \emptyset\}.
\end{aligned}$$

We remark that $\mathcal{U}_n = \mathcal{X}_n \cup \mathcal{Y}_n, \mathcal{X}_n \cap \mathcal{Y}_n = \mathcal{Z}_n$ and $\mathcal{Z}_n = \bigsqcup_i \mathcal{Z}_n^i$. Then the pair $(\mathcal{X}_n, \mathcal{Y}_n)$ forms an excision pair of $\mathcal{U}_n$ and the map $\mathcal{U}_n \to \mathcal{U}_{n+1}$ preserves the pairs. Thus we have the following exact sequence:

(2)

$$\cdots \to \varinjlim K_p(|\mathcal{Z}_n|) \to \varinjlim K_p(|\mathcal{X}_n|) \oplus \varinjlim K_p(|\mathcal{Y}_n|) \to \varinjlim K_p(|\mathcal{U}_n|) \to \varinjlim K_{p-1}(|\mathcal{Z}_n|) \to \cdots .$$

Since $\{\mathcal{U}_n\}_n$ forms an anti-Čech system of $X(G, \mathbb{P}, \mathcal{S})^{(0)}$, we have $\varinjlim K_*(|\mathcal{U}_n|) = KX_*(X(G, \mathbb{P}, \mathcal{S}))$. In this section, we compute $\varinjlim K_*(|\mathcal{X}_n|)$ and $\varinjlim K_*(|\mathcal{Y}_n|)$.

Lemma 2.6. *The inductive limit of $K_*(|\mathcal{X}_n|)$ is isomorphic to $KX_*(X(1))$.*

Proof. For $N \geq j+1 \geq 0$, we define that the subset $\mathcal{U}(N, j)$ of $\mathcal{U}(j)$ is made up of all columns $B((x,t), j) \in \mathcal{U}(j)$ which intersect with $X(N)$. We remark that $\mathcal{X}_n = \mathcal{U}(N_n, j_n)$. We define simplicial maps α_n, β_n, γ_n by

$$\begin{aligned}
&\alpha_n : \mathcal{U}(1, j_n) \to \mathcal{U}(N_n, j_n) && : B((x,t), j_n) \mapsto B((x,t), j_n),\\
&\beta_n : \mathcal{U}(N_n, j_n) \to \mathcal{U}(1, j_{n+1}) && : B((x,t), j_n) \mapsto \begin{cases} B((x,1), j_{n+1}) & (t \geq 1) \\ B((x,0), j_{n+1}) & (t = 0), \end{cases}\\
&\gamma_n : \mathcal{U}(N_n, j_n) \to \mathcal{U}(N_{n+1}, j_{n+1}) && : B((x,t), j_n) \mapsto B((x,t), j_{n+1}).
\end{aligned}$$

Clearly $\alpha_{n+1} \circ \beta_n$ and γ_n belong to the same contiguity class. Since two simplicial maps belonging to the same contiguity class define continuous maps which are homotopic [18, Lemma 5.5.2.], we have the following commutative diagram:

$$\begin{tikzcd}
K_*(|\mathcal{U}(1, j_n)|) \arrow[r, "\alpha_{n*}"] \arrow[d] & K_*(|\mathcal{U}(N_n, j_n)|) \arrow[dl, "\beta_{n*}"'] \arrow[d, "\gamma_{n*}"] \\
K_*(|\mathcal{U}(1, j_{n+1})|) \arrow[r, "\alpha_{n+1*}"] & K_*(|\mathcal{U}(N_{n+1}, j_{n+1})|).
\end{tikzcd}$$

It follows that $\varinjlim K_*(|\mathcal{U}(1,j_n)|) \cong \varinjlim K_*(|\mathcal{U}(N_n,j_n)|)$.

Let $\mathcal{U}(1,j_n)\cap X(1)$ denote the cover of $X(1)$ which consists of all $B\cap X(1)$ for $B\in \mathcal{U}(1,j_n)$. Then $\{\mathcal{U}(1,j_n)\cap X(1)\}_n$ forms an anti-Čech system of $X(1)$. Since $|\mathcal{U}(1,j_n)\cap X(1)| = |\mathcal{U}(1,j_n)|$, we have $KX_*(X(1)) = \varinjlim K_*(|\mathcal{X}_n|)$. $\square$

Lemma 2.7. *The inductive limit of $K_*(|\mathcal{Y}_n|)$ is trivial.*

Proof. For an integer $s\geq 0$, we define a simplicial map $q_{n,s}\colon \mathcal{Y}_n \to \mathcal{Y}_{n+1}$ by

$$q_{n,s}(B((x,t),j_n)) = \begin{cases} B((x,t),j_{n+1}) & \text{if } t\geq s,\\ B((x,s),j_{n+1}) & \text{if } t< s.\end{cases}$$

Clearly $q_{n,s}$ and $q_{n,s+1}$ are contiguous. Let $h_{n,s}\colon [s,s+1]\times|\mathcal{Y}_n|\to|\mathcal{Y}_{n+1}|$ be a proper homotopy between geometric realizations of $q_{n,s}$ and $g_{n,s+1}$. We define a proper map $q_n\colon \mathbb{R}_{\geq 0}\times|\mathcal{Y}_n|\to|\mathcal{Y}_{n+1}|$ by $q_n(\theta,x) = h_{n,\lfloor\theta\rfloor}(\theta,x)$, where $\theta\in\mathbb{R}_{\geq 0}$, $x\in|\mathcal{Y}_n|$, and $\lfloor\theta\rfloor$ denotes the largest integer not greater than θ. Then we have the following commutative diagram:

$$\begin{tikzcd} |\mathcal{Y}_n| \arrow[rr] \arrow[dr, hook] & & |\mathcal{Y}_{n+1}| \\ & \mathbb{R}_{\geq 0}\times|\mathcal{Y}_n| \arrow[ur, "q_n"'] & \end{tikzcd}$$

Here the horizontal arrow is the canonical map and the map $|\mathcal{Y}_n|\hookrightarrow\mathbb{R}_{\geq 0}\times|\mathcal{Y}_n|$ is given by the inclusion onto $\{0\}\times|\mathcal{Y}_n|$. Since $\mathbb{R}_{\geq 0}\times|\mathcal{Y}_n|$ is contractible (see [7, Remark 7.1.4]), the homomorphism $K_*(|\mathcal{Y}_n|)\to K_*(|\mathcal{Y}_{n+1}|)$ factors through zero. Therefore, $\varinjlim K_*(|\mathcal{Y}_n|) = 0$. $\square$

By the cluster axiom of K-homology (see [7, Definition 7.3.1]), we have $K_*(|\mathcal{Z}_n|)\cong \prod_{i\geq 1} K_*(|\mathcal{Z}_n^i|)$. Therefore we have the following exact sequence:

$$\cdots\to\varinjlim\prod_{i\geq 1}K_*(|\mathcal{Z}_n^i|)\to KX_*(X(1))\to KX_*(X(G,\mathbb{P},\mathcal{S}))\to\cdots. \tag{3}$$

We remark that $KX_*(X(1))\cong KX_*(G)$ since $X(1)$ and G are coarsely equivalent. In the next section, we will show $\varinjlim\prod_{i\geq 1}K_*(|\mathcal{Z}_n^i|)\cong\prod_{i\geq 1}KX_*(g_iP_{(i)})$ with the aid of finite universal spaces $\underline{E}P_1,\dots,\underline{E}P_k$.

3. Contractible models

In this section, we take $(G, \mathbb{P})$ in Theorem 1.1. Let $\underline{E}G$ be a finite G-simplicial complex which is a universal space for proper actions. For $r \in \{1, \dots, k\}$, let $\underline{E}P_r$ be a finite P_r-simplicial complex which is a universal space for proper actions. In the rest of this paper, we assume that all $\underline{E}P_r$ are embedded in $\underline{E}G$. We also assume that G is naturally embedded in the set of vertices of $\underline{E}G$ and $g_i P_{(i)}$ is embedded in $g_i \underline{E}P_{(i)}$. If $(G, \mathbb{P})$ satisfies conditions in Theorem 1.1, then we can take $\underline{E}G$ satisfying these conditions (see Appendix A). We take a finite subcomplex $\Delta \subset \underline{E}G$ containing a fundamental domain of $\underline{E}G$. We may assume that $\Delta \cap \underline{E}P_r$ contains a fundamental domain of $\underline{E}P_r$ for $r = 1, \dots, k$ without loss of generality.

Now, we introduce a contractible model of $X(G, \mathbb{P}, \mathcal{S})$. We define an embedding $\varphi_i \colon g_i \underline{E}P_{(i)} \times \{0\} \hookrightarrow \underline{E}G$ by $\varphi_i(x, 0) = x$.

A contractible model for $X(G, \mathbb{P}, \mathcal{S})$ is obtained by pasting $g_i \underline{E}P_{(i)} \times [0, \infty)$ to $\underline{E}G$ by φ_i for all $i \in \mathbb{N}$. Thus we can write it as follows:

$$EX(G, \mathbb{P}) = \underline{E}G \cup \bigcup_{i \in \mathbb{N}} (g_i \underline{E}P_{(i)} \times [0, \infty)).$$

Contractible models for $X(1), Y(1)$ and $\mathcal{H}(g_i P_{(i)}; \{1\})$ are also defined as follows:

$$\begin{aligned} EX(1) &= \underline{E}G \cup \bigcup_{i \in \mathbb{N}} (g_i \underline{E}P_{(i)} \times [0, 1]); \\ EY(1) &= \bigsqcup_{i \in \mathbb{N}} (g_i \underline{E}P_{(i)} \times [1, \infty)); \\ EZ^i &= g_i \underline{E}P_{(i)} \times \{1\}. \end{aligned}$$

We remark that $EX(G, \mathbb{P})$ admits a proper metric such that $EX(G, \mathbb{P})$ is coarsely equivalent to $X(G, \mathbb{P}, \mathcal{S})$, but it is neither of bounded geometry nor uniformly contractible, if $\mathbb{P}$ is not empty. Thus $EX(G, \mathbb{P})$ is not coarsening of $X(G, \mathbb{P}, \mathcal{S})$ in the sense of [17, Definition 2.4]. However $EX(G, \mathbb{P})$ is a “weakly coarsening” of $X(G, \mathbb{P}, \mathcal{S})$ in the following sense:

Proposition 3.1. *The coarse K-homology of $X(G, \mathbb{P}, \mathcal{S})$ can be computed by the contractible model, that is, $KX_*(X(G, \mathbb{P}, \mathcal{S})) \cong K_*(EX(G, \mathbb{P}))$.*

Proposition 3.1 is no direct consequence of [6, Proposition 3.8]. Our strategy is cutting off horoballs by Mayer-Vietoris arguments.

3.1. **Proof of Proposition 3.1.** We construct a locally finite cover $E\mathcal{U}_n$ of $EX(G,\mathbb{P})$ as follows: for $x \in g_iP_{(i)}$ and $j \geq 1$, the ball in $g_i\underline{E}P_{(i)}$ centered at x with the size j is

$$EB(x,j) = \bigcup y(\Delta \cap \underline{E}P_{(i)}) \tag{4}$$

where the union is taken over all $y \in g_iP_{(i)}$ such that $d_{\mathcal{S}}(x,y) \leq 2^j$. A contractible column centered at $(x,t) \in g_iP_{(i)} \times \mathbb{N}$ with the size j is

$$EB((x,t),j) = EB(x,t+j) \times [t,t+j].$$

For $x \in G$, a contractible column centered at $(x,0) \in G \times \{0\}$ with the size j is

$$EB((x,0),j) = \bigcup \left(y\Delta \cup \bigcup_{i\in\mathbb{N}} \left((y\Delta \cap g_i\underline{E}P_{(i)}) \times [0,j]\right)\right)$$

where the first union is taken over all $y \in G$ such that $d_{\mathcal{S}}(x,y) \leq 2^j$. We define that the cover $E\mathcal{U}_n$ of $EX(G,\mathbb{P})$ consists of all those columns $EB((x,t),j_n)$ for $(x,t) \in X(G,\mathbb{P},\mathcal{S})^{(0)}$. Taking subsequence if necessary, we define a simplicial map $E\mathcal{U}_n \to \mathcal{U}_{n+1}$ by $EB((x,t),j_n) \mapsto B((x,t),j_{n+1})$.

A partition of the unity gives a continuous map $h_n\colon EX(G,\mathbb{P}) \to |E\mathcal{U}_n|$. The composite of h_2 and $|E\mathcal{U}_2| \to |\mathcal{U}_3|$ induces a homomorphism $K_*(EX(G,\mathbb{P})) \to KX_*(X(G,\mathbb{P},\mathcal{S}))$.

Next, for each $i \in \mathbb{N}$, we construct an anti-Čech system $\{E\mathcal{Z}^i_n\}_n$ of EZ^i as follows: the cover $E\mathcal{Z}^i_n$ of EZ^i consists of all balls $EB(x,j_n)\times\{1\}$ for $x \in g_iP_{(i)}$. Then $\{E\mathcal{Z}^i_n\}_n$ forms an anti-Čech system.

We define a simplicial map $\mathcal{Z}^i_n \to E\mathcal{Z}^i_{n+1}$ by $B((x,s),j_n) \mapsto EB(x,j_{n+1}) \times \{1\}$. We also define a simplicial map $E\mathcal{Z}^i_n \to \mathcal{Z}^i_{n+1}$ by $EB(x,j_n)\times\{1\} \mapsto B((x,1),j_{n+1})$. Then we have a commutative diagram

$$\begin{tikzcd}
\prod_{i\in\mathbb{N}} K_*(|\mathcal{Z}^i_n|) \arrow[r] \arrow[d] & \prod_{i\in\mathbb{N}} K_*(|E\mathcal{Z}^i_{n+1}|) \arrow[dl] \arrow[d] \\
\prod_{i\in\mathbb{N}} K_*(|\mathcal{Z}^i_{n+2}|) \arrow[r] & \prod_{i\in\mathbb{N}} K_*(|E\mathcal{Z}^i_{n+3}|).
\end{tikzcd}$$

Hence $\varinjlim \prod_{i\in\mathbb{N}} K_*(|\mathcal{Z}^i_n|) \cong \varinjlim \prod_{i\in\mathbb{N}} K_*(|E\mathcal{Z}^i_n|)$. The partition of the unity gives a continuous map $h^i_n\colon EZ^i \to |E\mathcal{Z}^i_n|$ for i and $n \geq 1$. By the proof of [6, Proposition 3.8], taking a subsequence if necessary (not depending on i), the induced map $(h^i_n)_*\colon K_*(EZ^i) \to K_*(|E\mathcal{Z}^i_n|)$ is an isomorphism onto the image of the map $K_*(|E\mathcal{Z}^i_{n-1}|) \to K_*(|E\mathcal{Z}^i_n|)$. See

also [5, Lemma 7.11]. It follows that

$$\prod_{i\in\mathbb{N}} K_*(EZ^i) \cong \varinjlim \prod_{i\in\mathbb{N}} K_*(|E\mathcal{Z}^i_n|) \cong \varinjlim \prod_{i\in\mathbb{N}} K_*(|\mathcal{Z}^i_n|). \tag{5}$$

By arguments similar to that in the case of EZ^i, we can show the following isomorphism:

$$K_*(EX(1)) \cong \varinjlim K_*(|\mathcal{U}(1, j_n)|) = KX_*(X(1)). \tag{6}$$

By the Mayer-Vietoris sequence for $EX(G, \mathbb{P}) = EX(1) \cup EY(1)$, the exact sequence (3) and the fact that $K_*(EY(1)) = 0$, we have the following commutative diagram with two horizontal exact sequences:

$$\begin{array}{ccccccccc} \longrightarrow & \prod_{i\in\mathbb{N}} K_*(EZ^i(1)) & \longrightarrow & K_*(EX(1)) & \longrightarrow & K_*(EX(G,\mathbb{P})) & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & \varinjlim \prod_{i\in\mathbb{N}} K_*(|\mathcal{Z}^i_n|) & \longrightarrow & KX_*(X(1)) & \longrightarrow & KX_*(X(G,\mathbb{P},\mathcal{S})) & \longrightarrow & . \end{array} \tag{7}$$

By (5), (6) and the five lemma, all vertical maps are isomorphisms. This completes the proof of Proposition 3.1.

4. Coarse Mayer-Vietoris sequences

Higson, Roe and Yu [8] introduced a coarse Mayer-Vietoris sequence in the K-theory of the Roe algebras. It is used to prove a Lipschitz homotopy invariance of the K-theory of the Roe algebras [17, Theorem 9.8].

We first recall a notion of "excision pair" in coarse category. For a metric space M, a subspace A, and a positive number R, we denote by $\mathrm{Pen}(A; R)$ the R-neighbourhood of A in M, that is, $\mathrm{Pen}(A; R) = \{p \in M : d(p, A) \leq R\}$.

Definition 4.1. Let M be a proper metric space, and let A and B be closed subspaces with $M = A \cup B$. We say that $M = A \cup B$ is an ω-excisive decomposition, if for each $R > 0$ there exists some $S > 0$ such that

$$\mathrm{Pen}(A; R) \cap \mathrm{Pen}(B; R) \subset \mathrm{Pen}(A \cap B; S).$$

We summarize results in [8] (see also [12] and [13]) on coarse assembly maps and Mayer-Vietoris sequences as follows:

Theorem 4.2. *Suppose that $M = A \cup B$ is an ω-excisive decomposition. Then the following diagram is commutative and horizontal sequences are exact:*

$$\begin{tikzcd}
{} \arrow[r] & KX_p(A\cap B) \arrow[r] \arrow[d] & KX_p(A)\oplus KX_p(B) \arrow[r] \arrow[d] & KX_p(M) \arrow[r] \arrow[d] & KX_{p-1}(A\cap B) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & K_p(C^*(A\cap B)) \arrow[r] & K_p(C^*(A))\oplus K_p(C^*(B)) \arrow[r] & K_p(C^*(M)) \arrow[r] & K_{p-1}(C^*(A\cap B)) \arrow[r] & {}
\end{tikzcd}$$

Here vertical arrows are coarse assembly maps.

5. Proof of theorem 1.1

In this section, we give a proof of Theorem 1.1, which is divided into two parts. In the first part, we show inductively the coarse Baum-Connes conjecture for the space obtained by removing the first $n-1$ horoballs from $X(G,\mathbb{P},\mathcal{S})$. In the second part, we compute the coarse K-homology and the K-theory of the Roe algebra of G which is the intersection of a decreasing sequence of subspaces of $X(G,\mathbb{P},\mathcal{S})$.

5.1. **The first part.**

Notation 5.1. We introduce the following notations:

$$\begin{aligned}
X_n &= \Gamma \cup \bigcup_{i\geq n} \mathcal{H}(g_i P_{(i)});\\
X_\infty &= \bigcap_{n\geq 1} X_n;\\
EX_n &= \underline{E}G \cup \bigcup_{i\geq n} (g_i \underline{E}P_{(i)} \times [0,\infty))\\
EX_\infty &= \bigcap_{n\geq 1} EX_n.
\end{aligned}$$

We remark that $X_1 = X(G,\mathbb{P},\mathcal{S})$, $X_\infty = \Gamma$, $EX_1 = EX(G,\mathbb{P})$ and $EX_\infty = \underline{E}G$.

Since X_1 is δ-hyperbolic for some $\delta \geq 0$, by the result of Higson-Roe [6, Corollary 8.2], the coarse assembly map $\mu\colon KX_*(X_1) \to K_*(C^*(X_1))$ is an isomorphism. See Appendix B. In fact, by Proposition 3.1, the coarse assembly map

$$\mu\colon K_*(EX_1) \to K_*(C^*(X_1)) \tag{8}$$

is an isomorphism. By assumption and [6, Proposition 3.8], $\mu\colon K_*(g_n\underline{E}P_{(n)}) \to K_*(C^*(g_nP_{(n)}))$ is an isomorphism for all $n \geq 1$.

Lemma 5.2. *For any $n \geq 0$, the coarse assembly map $\mu_n\colon K_*(EX_n) \to K_*(C^*(X_n))$ is an isomorphism.*

Proof. We assume that μ_n is an isomorphism. Since $X_n = X_{n+1} \cup \mathcal{H}(g_n P_{(n)})$ is an ω-excisive decomposition, it follows from (coarse) Mayer-Vietoris sequences and the five lemma that μ_{n+1} is an isomorphism. $\square$

5.2. **The second part.** Let $(EX_n)^+$ denote the one-point compactification of EX_n. It is clear that $(EX_\infty)^+ = \bigcap_{n\in\mathbb{N}}(EX_n)^+$. By the Milnor exact sequence [7, Proposition7.3.4], we have

$$0 \to \varprojlim{}^1 K_{p+1}((EX_n)^+) \to K_p((EX_\infty)^+) \to \varprojlim K_p((EX_n)^+) \to 0. \tag{9}$$

Since the K-homology of EX_n is just the reduced K-homology of $(EX_n)^+$, we have $K_*((EX_n)^+) \cong K_*(EX_n) \oplus K_*(\{+\})$ where $\{+\}$ denotes a one-point space. This is also a direct consequence of an exact sequence [7, Definition7.1.1(b)]. Thus we can replace $K_*((EX_n)^+)$ in (9) by $K_*(EX_n)$.

Next, we consider the K-theory of the Roe algebras. Let H be a Hilbert space and $\rho\colon C_0(X_1) \to \mathfrak{B}(\mathrm{H})$ is an ample representation where $\mathfrak{B}(\mathrm{H})$ is the set of all bounded operators on H. The Roe algebra $C^*(X_1, \mathrm{H})$ is the norm closure of the algebra of locally compact, controlled operators on H (see [7, Definition 6.3.8]). The restriction $\rho\colon C_0(X_n) \to \mathfrak{B}(\overline{C_0(X_n)\mathrm{H}})$ gives an ample representation of $C_0(X_n)$. The Roe algebra $C^*(X_n, \overline{C_0(X_n)\mathrm{H}})$ can be naturally identified with a sub-C^*-algebra of $C^*(X_1, \mathrm{H})$, in fact, we have

$$C^*(X_n, \overline{C_0(X_n)\mathrm{H}}) = \{T \in C^*(X_1, \mathrm{H}) : \operatorname{supp} T \subset X_n \times X_n\}.$$

We abbreviate $C^*(X_n, \overline{C_0(X_n)\mathrm{H}})$ to $C^*(X_n)$. Now it is easy to see that $C^*(X_\infty) = \bigcap_{n\geq 1} C^*(X_n)$.

Phillips [16] studied the K-theory of the projective limit of C^*-algebras.

Proposition 5.3 ([16, Theorem 5.8(5)]). *The following sequence is exact.*

$$0 \to \varprojlim{}^1 K_{p+1}(C^*(X_n)) \to K_p(C^*(X_\infty)) \to \varprojlim K_p(C^*(X_n)) \to 0.$$

By Proposition 5.3 and (9), we have the following commutative diagram such that upper and lower horizontal sequences are exact:

$$\begin{tikzcd}
0 \arrow[r] & \varprojlim^1 K_{p+1}(EX_n) \arrow[r] \arrow[d] & K_p(EX_\infty) \arrow[r] \arrow[d] & \varprojlim K_p(EX_n) \arrow[r] \arrow[d] & 0. \\
0 \arrow[r] & \varprojlim^1 K_{p+1}(C^*(X_n)) \arrow[r] & K_p(C^*(X_\infty)) \arrow[r] & \varprojlim K_p(C^*(X_n)) \arrow[r] & 0.
\end{tikzcd}$$

By Lemma 5.2 and the five lemma, every vertical map is an isomorphism. This completes the proof of Theorem 1.1.

Remark 5.4. In the proof of Theorem 1.1, we use δ-hyperbolicity of the augmented space only for the first step of the induction in section 5.1 and the existence of a universal space $\underline{E}G$ mentioned in the beginning of Section 3.

Appendix A. A finite universal space for proper actions of a relatively hyperbolic group

In this appendix we prove the following (refer to [3, Theorem 0.1] on the case of torsion free groups):

Theorem A.1. *Let a countable group G be hyperbolic relative to a finite family of infinite subgroups $\mathbb{P}$. Suppose that every $P \in \mathbb{P}$ admits a finite P-simplicial complex which is a universal space for proper actions. Then G admits a finite G-simplicial complex which is a universal space for proper actions. In fact, G has a finite G-simplicial complex $\underline{E}G$ with an embedding $i\colon G \hookrightarrow \underline{E}G$ and each $P \in \mathbb{P}$ has a finite P-simplicial complex $\underline{E}P$ which is a subcomplex of $\underline{E}G$ such that $i(P) \subset \underline{E}P$.*

See [10] for universal spaces for proper actions.

Let a countable group G be finitely generated relative to a finite family of infinite subgroups $\mathbb{P}$. We denote the family of all left cosets by $\mathfrak{a} := \bigsqcup_{P\in\mathbb{P}} G/P$. We take a left invariant, proper metric d_G on G such that G is generated by $\{g \in G \mid d_G(e,g) \leq 1\} \cup \bigcup_{P\in\mathbb{P}} P$. We remark that $\{g \in G \mid d_G(e,g) \leq 1\}$ is a finite set.

Now we recall the definition of the augmented space $X(G,\mathbb{P},d_G)$ (see [4, Section 3] and also [9]). Its vertex set $V(G,\mathbb{P},d_G)$ is $G \sqcup \bigsqcup_{A\in\mathfrak{a}}(A \times \mathbb{N})$ where $\mathbb{N}$ is the set of positive integers. We often denote the subset $G \subset V(G,\mathbb{P})$ by $G \times \{0\}$. Also we often regard $A \in \mathfrak{a}$ as a subset $A \times \{0\}$ of $G \times \{0\}$. Its edge is either a vertical edge or a horizontal edge: a vertical edge is a pair $\{(a,t_1),(a,t_2)\} \subset A \times (\{0\} \sqcup \mathbb{N})$ such that $|t_1 - t_2| = 1$ for $A \in \mathfrak{a}$; a horizontal edge is a pair $\{(a_1,t),(a_2,t)\} \subset A \times \mathbb{N}$ such that $0 < d_G(a_1,a_2) \leq 2^t$ for $A \in \mathfrak{a}$ or a pair of $\{g_1,g_2\} \subset G$ such that $d_G(g_1,g_2) = 1$.

Since G is generated by $\{g \in G \mid d_G(e,g) \leq 1\} \cup \bigcup_{P\in\mathbb{P}} P$, the augmented space $X(G,\mathbb{P},d_G)$ is connected. This graph structure induces a metric on $V(G,\mathbb{P},d_G)$. When we consider for $P \in \mathbb{P}$, a left invariant proper metric $d_P := d_G|_{P\times P}$ on P, then $X(P,\{P\},d_P)$ is nothing but the full subgraph of $P \sqcup (P \times \mathbb{N})$ in $X(G,\mathbb{P},d_G)$. Moreover we can confirm that $X(P,\{P\},d_P)$ is an isometrically embedded subgraph of $X(G,\mathbb{P},d_G)$.

We consider the Rips complex $R_D(V(G,\mathbb{P},d_G))$ for a positive integer D. We denote the full subcomplexes of

$$V(G,\mathbb{P},d_G)_r = \bigsqcup_{A\in\mathfrak{a}}(A\times\{r,\dots\});$$
$$V(G,\mathbb{P},d_G)^R = G\sqcup\bigsqcup_{A\in\mathfrak{a}}(A\times\{1,\dots,R\});$$
$$V(G,\mathbb{P},d_G)_r^R = \bigsqcup_{A\in\mathfrak{a}}(A\times\{r,\dots,R\}) = V(G,\mathbb{P},d_G)_r\cap V(G,\mathbb{P},d_G)^R,$$

in $R_D(V(G,\mathbb{P},d_G))$ by $R_D(V(G,\mathbb{P},d_G))_r$, $R_D(V(G,\mathbb{P},d_G))^R$ and $R_D(V(G,\mathbb{P},d_G))_r^R$, respectively, where $r,R\in\mathbb{N}$ such that $r\le R$.

Remark A.2. If $r+D\le R$, then we have $R_D(V(G,\mathbb{P},d_G)) = R_D(V(G,\mathbb{P},d_G))_r\cup R_D(V(G,\mathbb{P},d_G))^R$ and $R_D(V(G,\mathbb{P},d_G))_r^R = R_D(V(G,\mathbb{P},d_G))_r\cap R_D(V(G,\mathbb{P},d_G))^R$.

G is hyperbolic relative to $\mathbb{P}$ if and only if $V(G,\mathbb{P},d_G)$ is δ-hyperbolic for some $\delta\ge 0$ (see [4, Theorem 3.25]). Since $V(G,\mathbb{P},d_G)$ is δ-hyperbolic, there exists some positive number D_δ such that for any $D\in\mathbb{N}$ such that $D\ge D_\delta$, the Rips complex $R_D(V(G,\mathbb{P},d_G))$ is contractible. Moreover we have the following:

Proposition A.3. *Let a countable group G be hyperbolic relative to a finite family of infinite subgroups $\mathbb{P}$. Suppose that $V(G,\mathbb{P},d_G)$ is δ-hyperbolic, where δ is a non-negative number. Then there exists some positive number D'_δ such that for any integer D such that $D\ge D'_\delta$, the first barycentric subdivision of the Rips complex $R_D(V(G,\mathbb{P},d_G))$ is a G-simplicial complex which is a universal space for proper actions.*

If $\mathbb{P}$ is empty on the above, then G is a hyperbolic group. The above for this case is known ([11]). Since arguments in the proof of [11, Theorem 1] can be applied to the above, we omit its proof.

Proof of Theorem A.1. We take a left invariant proper metric d_G on G such that G is generated by $\{g\in G\mid d_G(e,g)\le 1\}\cup\bigcup_{P\in\mathbb{P}}P$. We denote by d_P a left invariant proper metric $d_G|_{P\times P}$ on $P\in\mathbb{P}$.

Suppose that $V(G,\mathbb{P},d_G)$ is δ-hyperbolic. Then for every $P\in\mathbb{P}$, the vertex set $V(P,\{P\},d_P)$ is δ-hyperbolic because $X(P,\{P\},d_P)$ is an isometrically embedded subgraph of $X(G,\mathbb{P},d_G)$. We fix $D\in\mathbb{N}$ such that $D\ge D'_\delta$, where D'_δ is a constant in Proposition A.3. We take $P\in\mathbb{P}$ and $r,R\in\mathbb{N}$ such that $r+D\le R$. Also we take for every $P\in\mathbb{P}$, a finite P-simplicial complex $\underline{E}P$ which is a universal space for proper actions.

Since the first barycentric subdivision of $R_D(V(P,\{P\},d_P))_r$ is a P-simplicial complex which is a universal space for proper actions by Proposition A.3, we have a P-homotopy equivalent map $h_P : R_D(V(P,\{P\},d_P))_r \to \underline{E}P$. It follows from an equivariant version of simplicial approximation theorem (see [1, Exercise 6 for Chapter 1]) that there exist a natural number n and a P-simplicial map $f_P : R_D^{(n)}(V(P,\{P\},d_P))_r \to \underline{E}P$ which is P-homotopy equivalent to h_P where $R_D^{(n)}(V(P,\{P\},d_P))_r$ is the n-th barycentric subdivision of $R_D(V(P,\{P\},d_P))_r$. We can take n independently of P because $\mathbb{P}$ is a finite family. We consider mapping cylinders

$$(R_D^{(n)}(V(P,\{P\},d_P))_r^R \times [0,1]) \cup_{j_P} R_D^{(n)}(V(P,\{P\},d_P))_r;$$
$$(R_D^{(n)}(V(P,\{P\},d_P))_r^R \times [0,1]) \cup_{q_P} \underline{E}P,$$

whose pasting maps are

$$j_P : R_D^{(n)}(V(P,\{P\},d_P))_r^R \times \{1\} \ni (x,1) \mapsto x \in R_D^{(n)}(V(P,\{P\},d_P))_r;$$
$$q_P : R_D^{(n)}(V(P,\{P\},d_P))_r^R \times \{1\} \ni (x,1) \mapsto f_P(x) \in \underline{E}P,$$

respectively. Then the maps $id_{R_D^{(n)}(V(P,\{P\},d_P))_r^R}$ and f_P induce a map

$$\begin{aligned}\widetilde{f_P} :&(R_D^{(n)}(V(P,\{P\},d_P))_r^R \times [0,1]) \cup_{j_P} R_D^{(n)}(V(P,\{P\},d_P))_r \to \\ &(R_D^{(n)}(V(P,\{P\},d_P))_r^R \times [0,1]) \cup_{q_P} \underline{E}P,\end{aligned}$$

which is a P-homotopy equivalent map. In fact we can confirm that $\widetilde{f_P}$ is a P-homotopy equivalent map relative to $R_D^{(n)}(V(P,\{P\},d_P))_r^R \times \{0\}$. Now we construct two G-simplicial complex $R_D^{(n)}(V(G,\mathbb{P},d_G))_1$ and $R_D^{(n)}(V(G,\mathbb{P},d_G))_2$ as follows: First, $R_D^{(n)}(V(G,\mathbb{P},d_G))_1$ is obtained by, for every $P \in \mathbb{P}$, pasting G-equivariantly, $(R_D^{(n)}(V(P,\{P\},d_P))_r^R \times [0,1]) \cup_{j_P} R_D^{(n)}(V(P,\{P\},d_P))_r$, to $R_D^{(n)}(V(G,\mathbb{P},d_G))^R$ by the pasting map

$$R_D^{(n)}(V(P,\{P\},d_P))_r^R \times \{0\} \to R_D^{(n)}(V(P,\{P\},d_P))_r^R.$$

Second, $R_D^{(n)}(V(G,\mathbb{P},d_G))_2$ is obtained by, for every $P \in \mathbb{P}$, pasting G-equivariantly, $R_D^{(n)}(V(P,\{P\},d_P))_r^R \times [0,1] \cup_{q_P} \underline{E}P$ to $R_D^{(n)}(V(G,\mathbb{P},d_G))^R$ by the same pasting map. Then they are G-homotopy equivalent by the induced map by $id_{R_D^{(n)}(V(G,\mathbb{P},d_G))^R}$ and $\widetilde{f_P}$ for any $P \in \mathbb{P}$. Since $R_D^{(n)}(V(G,\mathbb{P},d_G))$ is clearly G-homotopic to $R_D^{(n)}(V(G,\mathbb{P},d_G))_1$ by Remark A.2, we have $R_D^{(n)}(V(G,\mathbb{P},d_G))$ is G-homotopic to $R_D^{(n)}(V(G,\mathbb{P},d_G))_2$. It follows from Proposition A.3 that $R_D^{(n)}(V(G,\mathbb{P},d_G))_2$ is a G-simplicial complex which is a universal space for proper actions. It is also clear that $R_D^{(n)}(V(G,\mathbb{P},d_G))_2$ is a finite G-simplicial complex by the construction. G is naturally embedded in $R_D^{(n)}(V(G,\mathbb{P},d_G))_2$.

$R_D^{(n)}(V(P,\{P\},d_P))_2$ is a subcomplex of $R_D^{(n)}(V(G,\mathbb{P},d_G))_2$ and is a finite universal P-simplicial complex with the natural embedding of P. □

Appendix B. The coarse Baum-Connes conjecture for hyperbolic metric spaces

Higson and Roe [6, Corollary 8.2] proved the coarse Baum-Connes conjecture for hyperbolic metric spaces. The following Proposition B.1 plays an important role in their proof.

Proposition B.1. *Let Y be a compact metric space and let $\mathcal{O}Y$ denote an open cone of Y. Then the coarsening map*

$$\mu\colon K_*(\mathcal{O}Y) \to KX_*(\mathcal{O}Y)$$

is an isomorphism.

Higson and Roe [6, Proposition 4.3] proved this proposition assuming that the dimension of Y is finite. Here we prove it without assuming that.

Proof. Any compact metric space can be embedded in the separable Hilbert space l_2. In fact, the stereographic projection gives an embedding in the unit ball of l_2. So we assume $Y \subset \{\mathbf{x} \in l_2 : \|\mathbf{x}\| = 1\}$. Then the open cone of Y is given by $\mathcal{O}Y = \{t\mathbf{x} \in l_2 : \mathbf{x} \in Y, t \in [0,\infty)\}$. For $I \subset (0,\infty)$, set

$$Y \times I = \{t\mathbf{x} \in l_2 : \mathbf{x} \in Y, t \in I\}.$$

Since Y is compact, for each $n \in N$, there exist $p_1^n, \dots, p_{a_n}^n \in Y \times \{n\}$ such that

$$\bigcup_{m=1}^{a_n} B(p_m^n, 1) \supset Y \times \{n\}. \tag{10}$$

Here $B(x,r)$ denotes a ball of radius r centered at x. Then we have

$$\bigcup_{m=1}^{a_n} B(p_m^n, 2) \supset Y \times [n-1, n+1].$$

For each $i \in \mathbb{N}$, we form a cover $\mathcal{U}_i$ of $\mathcal{O}Y$ as follows:

$$U_m^n(i) = B(p_m^n, 3^i) \cap \mathcal{O}Y,\ m = 1, \dots, a_n,$$
$$\mathcal{U}_i = \bigcup_{n\geq 1} \{U_1^n(i), \dots, U_{a_n}^n(i)\}.$$

It is clear that $\mathcal{U}_i$ is a locally finite cover and thus we obtain an anti-Čech system $\{\mathcal{U}_i\}_{i\geq 1}$. By the definition, it follows that

$$\bigcup_{n\geq 1}\bigcup_{m=1}^{a_n} B(p^n_m, 3^i) \subset \mathrm{Pen}(\mathcal{O}Y, 3^i).$$

Then the method used in the proof of [6, Proposition 4.3] can be applied to $\{\mathcal{U}_i\}_{i\geq 1}$. This completes the proof of Proposition B.1. □

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